It follows that the triangles PAB and PCD are isosceles with $PA = PB$ and $PC = PD$ (1).
Furthermore, since they are similar (congruent angles) we have
$\dfrac{PA}{AB} = \dfrac{PC}{CD} = \dfrac{PA+PC}{AB+CD} = \dfrac{AC}{AB+CD} = 1$. Thus, $PA = AB$ and $PC = CD$ (2).
From (1) and (2) we conclude that triangles PAB and PCD are equilateral.
Let $p = AB = PA = PB$, $q = CD = PC = PD$, $r = BC = AD$, $t = p + q + r$ and h the height of the trapezoid. Then we have
$\dfrac{p}{q} = \dfrac{ph}{qh} = \dfrac{2[ABC]}{2[ACD]} = \dfrac{12(2p+q+r)}{35(p+2q+r)} = \dfrac{12(p+t)}{35(q+t)}$ or $23pq = t(12q - 35p)$ (3)
Since the trapezoid is isosceles, it is cyclic, so by Ptolemy's Theorem we have
$pq + r^2 = (p+q)^2$ (4) or $pq = t(p+q-r)$ (5)
By (3) and (5) we obtain $58p + 11q = 23r$ (6)
Finally, applying the well-known formula $r = (s-a)\tan\dfrac{A}{2}$ in triangles ACD and BAC we have
$23 = 35 - 12 = \left(\dfrac{p+2q+r}{2} - r\right)\dfrac{\sqrt{3}}{3} - \left(\dfrac{2p+q+r}{2} - r\right)\dfrac{\sqrt{3}}{3} = \dfrac{q-p}{2}\cdot\dfrac{\sqrt{3}}{3}$, i.e.
$q - p = 46\sqrt{3}$ (7).
Solving the system of equations (4), (6) and (7) we find $p = 17\sqrt{3}$, $q = 63\sqrt{3}$, and $r = 73\sqrt{3}$.
Therefore the perimeter of trapezoid is $p + q + 2r = 226\sqrt{3}$.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Malik Sheykhov (student at the France-Azerbaijan University in Azerbaijan) and Talman Residli (student at Azerbaijan Medical University in Baku, Azerbaijan); David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer

- **5410:** *Proposed by Arkady Alt, San Jose, CA*

For the given integers $a_1, a_2, a_3 \geq 2$ find the largest value of the integer semiperimeter of a triangle with integer side lengths t_1, t_2, t_3 satisfying the inequalities $t_i \leq a_i,\ i = 1, 2, 3$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Without loss of generality, we assume that $a_1 \geq a_2 \geq a_3$. Let

$$T_1 = \{2, 3, \ldots, a_1\},\ T_2 = \{2, 3, \ldots, a_2\},\ T_3 = \{2, 3, \ldots, a_3\}$$

$$T = \{(t_1, t_2, t_3) : t_1 \in T_1,\ t_2 \in T_2,\ t_3 \in T_3\} \text{ and}$$

$$S = T \cap \{(t_1, t_2 t_3) : t_1, t_2, t_3 \text{ are the side lengths of a triangle}\}.$$

Let $L = \underset{(t_1,t_2,t_3)\in S}{\text{Maximum}} \dfrac{t_1+t_2+t_3}{2}$. We show that $L = \begin{cases} \dfrac{a_1+a_2+a_3}{2}, & \text{if } a_2 + a_3 > a_1 \\ a_2 + a_3 - \dfrac{1}{2}, & \text{if } a_2 + a_3 \leq a_1. \end{cases}$

Case 1: $a_2 + a_3 > a_1$

We have $(a_1, a_2, a_3) \in S$ and clearly $L = \dfrac{a_1 + a_2 + a_3}{2}$.

Case 2: $a_2 + a_3 \le a_1$

We have $(a_2 + a_3 - 1, a_2, a_3) \in S$ so that $L \ge a_2 + a_3 - \dfrac{1}{2}$. If $(t_1, t_2, t_3) \in T$ and $t_1 > a_2 + a_3 - 1$, then $(t_1, t_2, t_3) \notin S$. If $(t_1, t_2, t_3) \in T$ then $t_1 < a_2 + a_3 - 1$, then $\dfrac{t_1 + t_2 + t_3}{2} < \dfrac{(a_2 + a_3 - 1) + a_2 + a_3}{2} = a_2 + a_3 - \dfrac{1}{2}$. Hence, $L = a_2 + a_3 - \dfrac{1}{2}$ in this case.

This completes the solution.

Solution 2 by proposer

Let $s = \dfrac{t_1 + t_2 + t_3}{2}$. Since $t_i < s, i = 1, 2, 3$ then by the triangle inequality our problem becomes the following: Find the maximum of s for which there are positive integer numbers t_1, t_2, t_3 satisfying $\mathrm{t}_i \le \min\{a_i, s - 1), i = 1, 2, 3, t_1 + t_2 + t_3 = 2s$.

First note that $s \ge 3, t_i \ge 2, i = 1, 2, 3$. Indeed, since $t_i \le s - 1$ then $1 \le s - t_i, i = 1, 2, 3$ and therefore $t_1 = 2s - t_2 - t_3 = (s - t_2) + (s - t_3) \ge 2$. Cyclicly we obtian $t_2, t_3 \ge 2$ Hence, $2s \ge 6 \iff s \ge 3$.

Since $t_3 = 2s - t_1 - t_2,\ 2 \le t_3 \le \min\{a_3, s - 1\}$, then
$1 \le 2s - t_1 - t_2 \le \min\{a_3, s - 1\} \iff \max\{2s - t_1 - a_3, s + 1 - t_1\} \le t_2 \le 2s - 1 - t_1$, and therefore, we obtain the inequality for t_2, namely that

$$(\mathbf{1}) \quad \max\{2s - t_1 - a_3, s + 1 - t_1, 2\} \le t_2 \le \min\{2s - 1 - t_1, a_2, s - 1\}$$

with the conditions of solvability being:

$$(\mathbf{2}) \begin{cases} 2s - t_1 - a_3 & \le & s - 1 \\ 2s - t_1 - a_3 & \le & a_2 \\ s + 1 - t_1 & \le & a_2 \\ 2 & \le & 2s - 1 - t_1 \end{cases} \iff \begin{cases} s + 1 - a_3 & \le & t_1 \\ 2s - a_2 - a_3 & \le & t_1 \\ s + 1 - a_2 & \le & t_1 \\ t_1 & \le & 2s - 3 \end{cases}$$

Since $s - 1 \le 2s - 3$, then $(\mathbf{2})$ together with $2 \le t_1 \le \min\{a_1, s - 1\}$ gives us the bounds for t_1

$(\mathbf{3}) \max\{s + 1 - a_3, 2s - a_2 - a_3, s + 1 - a_2, 2\} \le t_1 \le= \min\{a_1, s - 1\}$.

Since $2 \le a_i, i = 1, 2, 3$ then $s + 1 - a_2 \le s - 1, s + 1 - a_3 \le s - 1$ and the solvability condition for $(\mathbf{3})$ becomes

$$s + 1 - a_3 \le a_1 \iff s \le a_1 + a_3 - 1, 2s - a_2 - a_3 \le a_1 \iff s \le \left\lfloor \frac{a_1 + a_2 + a_3}{2} \right\rfloor,$$

$$s + 1 - a_2 \le a_1 \iff s \le a_1 + a_2 - 1, 2s - a_2 - a_3 \le s - 1 \iff s \le +a_2 + a_3 - 1.$$

Thus, $s^* = \min\left\{\left\lfloor \dfrac{a_1 + a_2 + a_3}{2} \right\rfloor, a_1 + a_2 - 1, a_2 + a_3 - 1, a_3 + a_1 - 1\right\}$ is the largest integer value of the semiperimeter.

Solution 3 by Ed Gray, Highland Beach, FL

We consider several special cases:

a) If $a_1 = a_2 = a_3 = 2k$, we can equate $t_i = a_i$ for each i. The perimeter is then $6k$ and the semiperimeter is $3k$.

b) Suppose $a_1 = a_2 = a_3 = 2k+1$. We note that $a_1 + a_2 = 4k+2$ and $a_3 - 1 = 2k$. We define $t_1 = a_1,\ t_2 = a_2$ and $t_3 = a_3 - 1$.

c) Suppose that $a_1 = a_2$ and a_3 is larger than either one. In this case we set $t_1 = a_1$ and $t_2 = a_2$. It doesn't matter if a_1, a_2 are both even or both odd, $t_1 + t_2$ is even. We now have to avoid a potential problem. It must be true that $t_1 + t_2 \geq t_3$. Therefore, since if a_3 is large, we need to define $t_3 = a_3 - x$, where x is the integer which is the smallest such that $a_3 - x$ is even and $t_1 + t_2 > t_3$. Since $t_1 + t_2 + t_3$ is even, the semiperimeter is integral.

d) Suppose that $a_1 = a_2$ and a_3 is smaller than either one, in this case set $t_1 = a_1, t_2 = a_2$, so that $t_1 + t_2$ is even. If a_3=2, we let $t_3 = 2$. If $a_3 > 2$, but odd, we we set $t_3 = a_3 - 1$. Then $t_1 + t_2 + t_3$ equals the perimeter which is even and with an integer semiperimeter, and the triangle inequalities hold.

e) Finally, we have the general case: $a_1 < a_2 < a_3$. We set $t_1 = a_1, t_2 = a_2$. If $t_1 + t_2$ is even we need t_3 to be even. If a_3 is very far so that $a_1 + a_2 < a_3$, we let $t_3 = a_3 - x$, where x is the smallest integer which simultaneously makes $t_1 + t_2 + t_3$ even and $t_1 + t_2 > t_3$. If $t_1 + t_2$ is odd, we employ a similar calculation.

Solution 4 by Paul M. Harms, North Newton, KS

Suppose $a_1 \leq a_2 \leq a_3$. The largest perimeter would be $a_1 + a_2 + a_3$ where $t_i = a_i,\ i = 1, 2, 3$ provided that we have a triangle, i.e., $a_1 + a_2 > a_3$.

If $a_1 + a_2 > a_3$, and the perimeter is an even integer, then the largest value of an integer semiperimeter is $\dfrac{a_1 +_2 + a_3}{2}$.

If the perimeter is an odd integer, then a_3 must be at least 3 and we could use sides $t_1 = a_1, t_2 = a_2$ and $t_3 = a_3 - 1$. The largest integer semiperimeter for this case is $\dfrac{a_1 + a_2 + a_3 - 1}{2}$.

Now consider the case where $a_1 + a_2 \leq a_3$. A triangle with a maximum perimeter is when $t_1 = a_1,\ a_2 = t_2$, and $t_3 = a_1 + a_2 - 1$. Here $t_3 > a_1, a_2$ and the perimeter is the odd integer $2a_1 + 2a_2 - 1$. To get the largest integer semiperimeter we could use $t_1 = a_1,\ t_2 = a_2$ and $t_3 = a_1 + a_2 - 2$ which has $a_1 + a_2 - 1$ as the largest integer semiperimeter.

Also solved by Jeremiah Bartz and Timothy Prescott, University of North Dakota, Grand Forks, ND; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

- **5411:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be real valued positive sequences with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a \in R_+^*$

If $\lim_{n\to\infty} (n(a_n - a)) = b \in R$ and $\lim_{n\to\infty} (n(b_n - a)) = c \in R$ compute

$$\lim_{n\to\infty} \left(a_{n+1} \sqrt[n+1]{(n+1)!} - b_n \sqrt[n]{n!} \right).$$

Note: R_+^* means the positive real numbers without zero.